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Electromagnetic moments for arbitrary spin

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Abstract. A form of the Dirac equations for arbitrary spin, with minimal electromagnetic coupling, is considered in the non-relativistic limit to first order in e. An approximate expansion of the Hamiltonian in powers of $1/m$ is obtained. The electric quadrupole moment, spin–orbit coupling and Darwin term are evaluated, and for spin 1 are found to differ from those obtained on Proca theory. A higher approximation is calculated for static magnetic and electric fields.

1. Introduction

It has been shown (Dowker 1966 a) that minimal electromagnetic couplings may be introduced consistently into the conventional equations for a massive particle of spin j (Dirac 1936). The analogue of the Klein–Gordon equation to first order in e appears as†

$$\left\{D_\mu D^\mu + m^2 - \frac{e}{2j} F_{\mu\nu} J^{\mu\nu}(j)\right\}\psi(x) = 0 \tag{1}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor, ψ is a $(2j+1)$-component spinor and $J_{\mu\nu}(j)$ are the generators of the homogeneous Lorentz group in the $(j, 0)$ representation.

This equation may be written in Hamiltonian form by a doubling of the representation space (Dowker 1966 b). Defining a $2(2j+1)$-component vector by

$$\varphi = \frac{1}{2}\begin{pmatrix} \psi + \dfrac{i}{m} D_0 \psi \\ \psi - \dfrac{i}{m} D_0 \psi \end{pmatrix}$$

we may write equation (1) as

$$i\frac{\partial \varphi}{\partial t} = \mathscr{H}\varphi$$

where

$$\mathscr{H} = \tau_3 m + eV - \frac{\tau_3 + i\tau_2}{2m}\left\{D^2 + \frac{e}{j}\mathbf{J}.(\mathbf{H} - i\mathbf{E})\right\}. \tag{2}$$

Here $D^2 = (\nabla - ie\mathbf{A})^2$ and V is the scalar potential (the charge on the particle is $+e$). The matrices are given by

$$\tau_1 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 0 & -i\mathbf{1} \\ i\mathbf{1} & 0 \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

where $\mathbf{1}$ is the $(2j+1) \times (2j+1)$ unit matrix.

The Coulomb scattering cross section for arbitrary spin has been calculated on the basis of equation (1) (Dowker 1966 b). The result for spin $\frac{1}{2}$ agrees with Mott's formula, but for spin 1 differs from that obtained on Proca theory (see references in Dowker 1966 b, § 5). The question thus arises as to the difference between a 'tensor' particle and a 'spinor' particle, presumably to be found in their multipole structures.

In the present paper we consider the non-relativistic limit of equation (2) in order to evaluate the multipole structure of the 'spinor' particle.

† Here $D_\mu = \partial_\mu - ieA_\mu$, $A^\mu = (\mathbf{A}, V)$ and $\hbar = c = 1$.

2. Non-relativistic limit

To find the non-relativistic limit of equation (2) we cast the wave equation into canonical form (Foldy 1956). The free-particle Hamiltonian is diagonalized by the transformation (Foldy and Wouthuysen 1950, Case 1954)

$$\mathscr{H}' = e^{iS}\mathscr{H}\, e^{-iS} \tag{3}$$

where

$$S = -\tfrac{1}{2}i\tau_1 \tanh^{-1}\left(\frac{p^2/2m}{m^2+p^2/2m}\right).$$

Thus in the non-interacting case

$$\mathscr{H}' = \tau_3(m^2+p^2)^{1/2}$$

so that each sign of the energy can be represented by a $(2j+1)$-component wave function.

In the presence of a general electromagnetic field we cannot perform a closed transformation on the Hamiltonian to eliminate the 'odd' terms (defined as those containing τ_1 or τ_2). We can, however, by a method of successive approximations, eliminate the 'odd' operators to any desired order in the inverse mass (Foldy and Wouthuysen 1950, Case 1954). As we are seeking the non-relativistic limit, this will be sufficient. It can be checked that the time-dependent transformation

$$\mathscr{H}' = e^{iS}\left(\mathscr{H} - i\frac{\partial}{\partial t}\right) e^{-iS} \tag{4}$$

where

$$S = \frac{i\tau_1}{4m^2}\left\{D^2 + \frac{e}{j}\mathbf{J}.(\mathbf{H}-i\mathbf{E})\right\} + \frac{\tau_2 e}{8m^3}\left\{(\nabla.\mathbf{E}+\mathbf{E}.\nabla) - \frac{i}{j}\mathbf{J}.\left(\frac{\overline{\partial\mathbf{H}}}{\partial t} - i\frac{\overline{\partial\mathbf{E}}}{\partial t}\right)\right\}$$

will perform the required eliminations on the interacting Hamiltonian† correct to order $1/m^2$.

The resulting wave equation is

$$i\frac{\partial\varphi'}{\partial t} = \mathscr{H}'\varphi'$$

where

$$\mathscr{H}' = \tau_3 m + eV - \frac{\tau_3}{2m}\left\{D^2 + \frac{e}{j}\mathbf{J}.(\mathbf{H}-i\mathbf{E})\right\} + \mathrm{O}\left(\frac{1}{m^3}\right). \tag{5}$$

We must now consider the non-Hermitian term $ie\tau_3\mathbf{J}.\mathbf{E}/2jm$, which occurs with an 'even' operator in equation (5) and cannot be removed by the foregoing method. We can, however, make use of a special property of the Hamiltonian. Performing the transformation (4), where now $S = (\tau_3/2jm)\mathbf{J}.\mathbf{D}$, we obtain

$$\mathscr{H}'' = \tau_3 m + eV - \frac{\tau_3}{2m}\left(D^2 + \frac{e}{j}\mathbf{J}.\mathbf{H}\right) + \frac{e}{8j^2m^2}[\mathbf{J}.\mathbf{E}, \mathbf{J}.\nabla]$$
$$+ \frac{i}{4jm^2}\left[\left(D^2+\frac{e}{j}\mathbf{J}.\mathbf{H}\right), \mathbf{J}.\mathbf{D}\right] + \mathrm{O}\left(\frac{1}{m^3}\right). \tag{6}$$

The Hamiltonian‡ is now diagonalized and Hermitian to order $1/m^2$, except for the commutator involving the magnetic quantities. Part of this commutator can be written as

$$[D^2, \mathbf{J}.\mathbf{D}] = \mathbf{D}.[\mathbf{D}, \mathbf{J}.\mathbf{D}] + [\mathbf{D}, \mathbf{J}.\mathbf{D}].\mathbf{D}. \tag{7}$$

† The bars over the time derivatives indicate that these operate only on the fields:

$$\frac{\overline{\partial\mathbf{H}}}{\partial t} = \left[\frac{\partial}{\partial t}, \mathbf{H}\right].$$

‡ The fourth term indicates a magnetic moment of $\mu_z = e/2m$.

Since

$$[\mathbf{D}, \mathbf{J} \,.\, \mathbf{D}] = -ie[\nabla, \mathbf{J} \,.\, \mathbf{A}] + ie[\mathbf{J} \,.\, \nabla, \mathbf{A}] = -ie\mathbf{J} \times \mathbf{H}$$

equation (7) becomes, to first order in e,

$$[D^2, \mathbf{J} \,.\, \mathbf{D}] = -ie(\nabla \,.\, \mathbf{J} \times \mathbf{H} + \mathbf{J} \times \mathbf{H} \,.\, \nabla) = ie\mathbf{J} \,.\, (\nabla \times \mathbf{H} - \mathbf{H} \times \nabla).$$

On the other hand, it will be shown in equation (8) below that the remaining part of the commutator can be expanded as

$$[\mathbf{J} \,.\, \mathbf{H}, \mathbf{J} \,.\, \nabla] = -\tfrac{1}{2}[J_i, J_j]_+ \overline{\nabla_i H_j} + \tfrac{1}{2}i\mathbf{J} \,.\, (\mathbf{H} \times \nabla - \nabla \times \mathbf{H}).$$

Hence the total non-Hermitian part of (6) is given by

$$\frac{i}{4jm^2}\left[\left(D^2 + \frac{e}{j}\mathbf{J}.\mathbf{H}\right), \mathbf{J}.\mathbf{D}\right] = \frac{e}{8j^2m^2}\{(2j-1)\mathbf{J}.(\mathbf{H} \times \nabla - \nabla \times \mathbf{H}) - i[J_i, J_j]_+ \overline{\nabla_i H_j}\}.$$

It can be seen that this expression vanishes as required for spin $\frac{1}{2}$ (if we bear in mind that $\operatorname{div} \mathbf{H} = 0$). For higher spins, however, it does not vanish.

The other commutator in equation (6) (which is Hermitian) can be expanded as follows:

$$[\mathbf{J} \,.\, \mathbf{E}, \mathbf{J} \,.\, \nabla] = J_i J_j E_i \nabla_j - J_j J_i \nabla_j E_i.$$

Since

$$J_i J_j = \tfrac{1}{2}[J_i, J_j]_+ + \tfrac{1}{2}iJ_k \epsilon_{ijk}$$

this becomes

$$\begin{aligned}[\mathbf{J} \,.\, \mathbf{E}, \mathbf{J} \,.\, \nabla] &= -\tfrac{1}{2}[J_i, J_j]_+(\nabla_j E_i - E_i \nabla_j) + \tfrac{1}{2}iJ_k\epsilon_{ijk}(E_i \nabla_j + \nabla_j E_i) \\ &= -\tfrac{1}{2}[J_i, J_j]_+ \overline{\nabla_i E_j} + \tfrac{1}{2}i\mathbf{J} \,.\, (\mathbf{E} \times \nabla - \nabla \times \mathbf{E}). \qquad (8)\end{aligned}$$

We may therefore write this term of equation (6) as

$$\begin{aligned}\frac{e}{8j^2m^2}[\mathbf{J}.\mathbf{E}, \mathbf{J}.\nabla] = \frac{e}{8j^2m^2}\{&-\tfrac{1}{6}(3[J_i, J_j]_+ - 2\delta_{ij}\mathbf{J}^2)\overline{\nabla_i E_j} - \tfrac{1}{3}\mathbf{J}^2\overline{\nabla \,.\, \mathbf{E}} \\ &+ \tfrac{1}{2}i\mathbf{J} \,.\, (\mathbf{E} \times \nabla - \nabla \times \mathbf{E})\}. \qquad (9)\end{aligned}$$

The last three terms in this equation are the generalization to arbitrary spin of the Darwin and spin–orbit coupling terms. For spin $\frac{1}{2}$ they reduce to the usual terms familiar from the theory of the electron†. The remaining expression is the electric quadrupole interaction‡, the quadrupole tensor being given by

$$Q_{ij} = \frac{e}{8j^2m^2}(3[J_i, J_j]_+ - 2\delta_{ij}\mathbf{J}^2).$$

This tensor, of course, vanishes for spin $\frac{1}{2}$, but is non-zero for higher spins. In the form conventionalized by Ramsey (1953, p. 17) the quadrupole strength Q is defined by

$$Q_{ij} = \frac{eQ}{2j(2j-1)}(3[J_i, J_j]_+ - 2\delta_{ij}\mathbf{J}^2).$$

Thus our particle has an electric quadrupole moment of strength

$$Q = \frac{2j-1}{4jm^2}.$$

† See, for example, Bjorken and Drell (1964) and Schweber (1961). The first reference gives the spin–orbit coupling in a form which is not manifestly Hermitian.

‡ The interaction energy is of the form (Ramsey 1953, p. 16, Preston 1965)

$$\mathscr{H}_Q = -\tfrac{1}{6}Q_{ij}\overline{\nabla_i E_j}.$$

For particles of spin 1 an approximate non-relativistic Hamiltonian has been obtained by Case (1954). Performing a series of Foldy–Wouthuysen transformations on the Sakata–Taketani equations, we find that the contribution of order $1/m^2$ vanishes. A treatment using the most general first-order Proca Lagrangian (Young and Bludmen 1961, unpublished) leads to the following expressions for magnetic dipole and spin–orbit interactions:

$$\mathscr{H}_1 = -\frac{e}{2m} g\mathbf{J}\,.\,\mathbf{H}, \qquad \mathscr{H}_2 = \frac{ie}{4m^2}(g-1)\mathbf{J}\,.\,(\mathbf{E}\times\nabla - \nabla\times\mathbf{E}).$$

Here g is an arbitrary parameter. Choosing $g = 1$ so that $\mathscr{H}_1$ agrees with our result, we see again that the spin–orbit coupling vanishes (although a further arbitrariness remains in the quadrupole moment).

These results are to be compared with our equations (6) and (9). It can be seen that the electromagnetic structure of the 'spinor' particle is quite different from that of the Proca particle (even in the most general form), presumably because of the different effects of introducing minimal interactions in the two cases (see Taub 1939).

3. Static magnetic and electric fields

For the special case of a static magnetic field an exact decomposition into positive and negative energy states is possible (Case 1954, Dowker 1966 b). Setting $\mathbf{E} = V = 0$ in equation (2), we make the transformation (3), where now

$$\exp(iS) = \frac{1}{2(m\omega)^{1/2}}\{(\omega+m)+\tau_1(\omega-m)\}$$

and

$$\omega = \left(m^2 - D^2 - \frac{e}{j}\mathbf{J}\,.\,\mathbf{H}\right)^{1/2}.$$

The resulting Hamiltonian is

$$\mathscr{H}' = \tau_3\omega$$

which can be expanded as

$$\mathscr{H}' = \tau_3 m - \tau_3\left(\frac{D^2}{2m} + \frac{D^4}{8m^3} + \ldots\right) - \tau_3\frac{e}{j}\left(\frac{1}{2m}\mathbf{J}\,.\,\mathbf{H} + \frac{1}{8m^3}[\nabla^2, \mathbf{J}\,.\,\mathbf{H}]_+ + \ldots\right)$$

displaying the relativistic corrections to kinetic energy and magnetic dipole interaction. Clearly there are no higher magnetic multipoles for any spin.

For a particle in an electrostatic field ($\mathbf{H} = \mathbf{A} = 0$ in equation (2)) the appropriate transformation cannot be obtained in closed form. We use the method of successive approximations (Case 1954). The transformation (4), where $S = -i\tau_3 O/2m$ and O is the lowest-order 'odd' term in the Hamiltonian, will eliminate O to this order. Performing a sequence of such transformations, we obtain, correct to order $1/m^4$, the diagonalized Hamiltonian

$$\mathscr{H}' \simeq \tau_3 m + eV - \frac{\tau_3}{2m}\left(\nabla^2 - \frac{ie}{j}\mathbf{J}\,.\,\mathbf{E}\right)\left\{1 + \frac{1}{4m^2}\left(\nabla^2 - \frac{ie}{j}\mathbf{J}\,.\,\mathbf{E}\right)\right\}$$
$$+\frac{e}{32m^4}(\nabla^4 V - 2\nabla^2 V\nabla^2 + V\nabla^4). \tag{10}$$

The last expression can be written as

$$\frac{e}{32m^4}[\nabla^2, [\nabla^2, V]] = \frac{e}{32m^4}\overline{\nabla^4 V}$$

and is simply a higher-order (Hermitian) correction to the electrostatic energy.

In order to remove the non-Hermitian terms from equation (10), we perform a further transformation using $S = \tau_3 \mathbf{J}.\nabla/2jm$. The result is

$$\begin{aligned}\mathscr{H}'' \simeq\ & \tau_3 m + eV - \frac{\tau_3}{2m}\nabla^2 + \frac{e}{8j^2m^2}[\mathbf{J}.\mathbf{E},\ \mathbf{J}.\nabla] - \frac{\tau_3}{8m^3}\nabla^4 \\ & + \frac{ie\tau_3}{8jm^3}[\nabla^2,\ \mathbf{J}.\mathbf{E}]_+ - \frac{ie\tau_3}{24j^3m^3}[\mathbf{J}.\nabla,\ [\mathbf{J}.\nabla,\ \mathbf{J}.\mathbf{E}]] \\ & + \frac{e}{32m^4}\overline{\nabla^4 V} + \frac{e}{16j^2m^4}[\nabla^2,\ [\mathbf{J}.\mathbf{E},\ \mathbf{J}.\nabla]]_+ \\ & + \frac{e}{128j^4m^4}[\mathbf{J}.\nabla,\ [\mathbf{J}.\nabla,\ [\mathbf{J}.\nabla,\ \mathbf{J}.\mathbf{E}]]]. \qquad (11)\end{aligned}$$

The second term from the end is Hermitian, and by comparison with equation (9) is clearly a relativistic correction to the ordinary quadrupole, Darwin and spin–orbit interactions. The terms of order $1/m^3$ are non-Hermitian, except for the kinetic energy contribution $-\tau_3\nabla^4/8m^3$. Presumably these non-Hermitian terms are removable, but we are unable to find the appropriate transformation. However, the exponent of any such transformation will be of order $1/m^3$, and provided it is simply a differential operator it will commute with the term of order $1/m$ in the Hamiltonian. Hence such a transformation will make no contribution to order $1/m^4$. We therefore examine the remaining (Hermitian) term of this order:

$$\begin{aligned}[\mathbf{J}.\nabla,\ [\mathbf{J}.\nabla,\ [\mathbf{J}.\nabla,\ \mathbf{J}.\mathbf{E}]]] =\ & J_iJ_jJ_kJ_l\nabla_i\nabla_j\nabla_kE_l - J_lJ_kJ_jJ_iE_l\nabla_k\nabla_j\nabla_i \\ & + J_iJ_lJ_kJ_j\nabla_iE_l\nabla_k\nabla_j - J_jJ_kJ_lJ_i\nabla_j\nabla_kE_l\nabla_i \\ & + J_jJ_lJ_kJ_i\nabla_jE_l\nabla_k\nabla_i - J_iJ_kJ_lJ_j\nabla_i\nabla_kE_l\nabla_j \\ & + J_kJ_lJ_jJ_i\nabla_kE_l\nabla_j\nabla_i - J_iJ_jJ_lJ_k\nabla_i\nabla_jE_l\nabla_k. \qquad (12)\end{aligned}$$

From this expression we hope to obtain the electric multipole interaction of order 4, together with the analogue of spin–orbit coupling terms. The appropriate derivative is given by

$$\begin{aligned}\overline{\nabla_i\nabla_j\nabla_kE_l} &= [\nabla_i,\ [\nabla_j,\ [\nabla_k,\ E_l]]] \\ &= \nabla_i\nabla_j\nabla_kE_l + \nabla_iE_l\nabla_k\nabla_j + \nabla_jE_l\nabla_k\nabla_i + \nabla_kE_l\nabla_j\nabla_i \\ &\quad - E_l\nabla_k\nabla_j\nabla_i - \nabla_j\nabla_kE_l\nabla_i - \nabla_i\nabla_kE_l\nabla_j - \nabla_i\nabla_jE_l\nabla_k.\end{aligned}$$

On the other hand, the leading term in the multipole tensor will be a totally symmetrized product:

$$Q_{ijkl} = \underset{i,j,k,l}{\mathrm{S}}(J_iJ_jJ_kJ_l).$$

We can permute our products into this form by means of relations such as

$$\begin{aligned}4J_iJ_jJ_kJ_l =\ & J_iJ_jJ_kJ_l + J_iJ_jJ_lJ_k + J_iJ_lJ_jJ_k + J_lJ_iJ_jJ_k \\ & + i\epsilon_{klm}(3J_iJ_jJ_m + 2J_jJ_mJ_i + J_mJ_iJ_j)\end{aligned}$$

whence, symmetrizing over the indices i, j, k,

$$24J_iJ_jJ_kJ_l = \underset{i,j,k,l}{\mathrm{S}}(J_iJ_jJ_kJ_l) + \underset{i,j,k}{\mathrm{S}}\, i\epsilon_{klm}(3J_iJ_jJ_m + 2J_jJ_mJ_i + J_mJ_iJ_j).$$

The extra terms generated in this equation, when substituted into equation (12), are clearly in the nature of 'spin–orbit' couplings (cf. equation (8)).

A series of such permutations would enable us to extract from equation (12) the leading product of the multipole interaction:

$$\mathscr{H}_4 = Q_{ijkl}\overline{\nabla_i \nabla_j \nabla_k E_l}.$$

The complete multipole tensor, however, will be more complicated than this, involving products of Kronecker deltas; we should expect, for example, that it would vanish for $j<2$.

Further considerations along these lines would lead us too far astray. We merely note that the 'spinor' particle will evidently have electric moments of arbitrary order, subject to the invariance and group theory restrictions†.

4. Conclusion

The Hamiltonian for a 'spinor' particle in a general electromagnetic field has been calculated in the non-relativistic limit to order $1/m^2$. The terms of this order for spin 1 are found to differ from those obtained on Proca theory, even in the general form of Young and Bludmen (1961, unpublished). The electric quadrupole moment, spin–orbit coupling and Darwin term have been evaluated for arbitrary spin. Magnetic multipoles beyond the first order have been seen to vanish. Electric multipoles, however, exist for any accessible order and have been outlined to order $1/m^4$.

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† Group theory and parity (or time-reversal symmetry) restrict the order to $\lambda = 2n \leqslant 2j$, where n is an integer.